Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT®

ELSEVIER International Journal of Approximate Reasoning 33 (2003) 287–301

INTERNATIONAL JOURNAL OF APPROXIMATE REASONING

www.elsevier.com/locate/ijar

The reasonableness of necessity

Paul Snow *

P.O. Box 6134, Concord, NH 03303-6134 USA

Received 1 August 2002; accepted 1 December 2002

Abstract

Prospects for normative consensus between probabilists and advocates of the necessity calculus are nuanced. Necessity syntactically restates some probability distributions' orderings and satisfies Cox's "probabilistic" reasonableness standards, as possibility is now known to do. Used as a possibilistic tie breaker, necessity both restates probabilistic orderings and brings possibility closer to de Finetti's quasi-additive standard. Nevertheless, variations in necessity's credal orderings when beliefs change strain consensus. Moreover, in domains like the evaluation of scientific hypotheses, mathematical conjectures, and judicial findings, the negation of a hypothesis, needed to define necessity, may be ill-specified. Necessity may be less helpful to possibility in those domains, where professions of "belief" sometimes reflect not only credibility but also utilitarian or aesthetic preferences. Unbroken ties allow possibility to express both credibility and preference simultaneously.

Keywords: Necessity; Possibility; Atomic bound probabilities; Belief representation; Preference representation; Polya's heuristic

1. Introduction

The possibility calculus [1] evaluates disjunctions according to the rule

$$\Pi(A \vee B) = \max(\Pi(A), \Pi(B))$$

* Tel.: +1-603-228-2905.
E-mail address: paulusnix@cs.com (P. Snow).

0888-613X/03/$ - see front matter
doi:10.1016/S0888-613X(03)00019-7

By convention, a tautology has a possibility of unity. Under uncertainty, at least one non-tautological sentence must also have a possibility of unity, so that the *max* rule yields the conventional value for the tautology. A contradiction has the value of zero, and that value is typically reserved for known falsehoods. Throughout the paper, and regardless of the calculus being discussed, it is assumed that the domain of sentences being evaluated is finite.

Possibility has sometimes been portrayed as fundamentally different from probabilistic approaches to uncertainty. However, there were well-known similarities [2] among possibility, operations involving probability intervals, and the Dempster–Shafer calculus, which was itself originally proposed as an innovation in Bayesian technique [3].

More recently, closer connections between probability and possibility have been uncovered. These concern the similar semantics for default reasoning provided by the two calculi, their satisfaction of Cox's [4] criteria for credal reasonableness, and the ability to translate syntactically between all possibilistic and some probabilistic orderings.

Necessity is defined as

$$N(A) \equiv 1 - \Pi(\neg A)$$

Necessity may be used alone, or else together with possibility for what will be called "tie breaking." For example, if A is a tautology, and B is an uncertain sentence containing an atom whose possibility is unity, then the possibilities tie at unity, but

$$N(A) > N(B)$$

The ability to distinguish tautologies from uncertainties has obvious appeal.

The next section reviews results about the reasonableness of the possibility calculus from a probabilist perspective, including some new clarifications which reflect the views of non-Bayesian probabilists. Against that background, the paper then examines necessity.

By itself, necessity possesses "probabilistic" features similar to those of possibility. Some of its tie-breaking behavior in concert with possibility also has normative appeal for probabilists. Many probabilists dissent, however, from an aspect of necessity's performance, both alone and as a tie breaker, during belief revision.

In the later sections, the paper considers uncertain domains where properties different from those offered by necessity might be desirable. In these domains, the range of alternative hypotheses, and hence the "not A" upon which the definition of necessity depends, may be unknown. Hypothesis evaluation within these domains reflects not just relative credibility, but also preferences among competing hypotheses. Possibility can accomplish such a dual-purpose representation, provided that possibilistic ties are left unbroken.

2. The probabilistic reasonableness of possibility

Orderings of sentences based on possibility values respect the constraint

$$\text{if } A \Rightarrow B, \text{ then } B \text{ is ranked no less than } A$$

This behavior is a widely accepted criterion of intuitive credibility ordering identified by Łukasiewicz [5] and revived by Sugeno [6]. Probability is another familiar example of a Łukasiewicz–Sugeno calculus, as are the ordinary Boolean logic and Łukasiewicz' multivalued logics, the ancestors of what are now more commonly known as "fuzzy" logics.

Considerable similarity between possibility and probability prevails in connection with conditional logics and default reasoning [7]. A particular class of probability distributions faithfully emulates the behavior of the default entailment connective [8].

These *atomic bound* probabilities solve simultaneous constraint systems whose typical constraint is

$$p(x) > \sum_{\text{all atoms } y \text{ inferior to } x} p(y)$$

for each atom x among the sentences under consideration. Between distinct atoms, either one is inferior to the other, or else there is no constrained order between them, in which case they share the same lower bound constraint expression.

For those atoms which have no inferior atoms, their constraint is simply that of being strictly positive. Thus all atoms are ordered strictly ahead of falsehoods, an assumption that will always hold throughout this paper, whatever the calculus. The only other constraint in an atomic bound system is total probability, that the sum of the atomic probabilities is unity.

Strict atomic bound systems are those in which every pair of atoms is constrained to be ordered. An example solution over the five exclusive and exhaustive sentences a through e is:

$$p(a) = 16/31, \quad p(b) = 8/31, \quad p(c) = 4/31,$$
$$p(d) = 2/31, \quad p(e) = 1/31$$

For all mutually exclusive sentences A and B, strict atomic bound probabilities follow the possibilistic *max* rule, that is, $p(A) > p(B)$ just when the highest probability atom in $A \vee B$ is in A and not in B.

As is well known, possibility distributions also achieve emulation of the default rules. Of particular interest are the *linear* possibilities [9], those where each atomic sentence has a distinct possibility value. Linear possibilities apply the same ordinal rule about highest possibility atoms as strict atomic bound probabilities. It is easy to see that when the calculi agree about the order of the atoms, they are also in ordinal agreement for all other mutually exclusive sentences.

Agreement for exclusive sentences entails a more extensive relationship between the calculi. An ordinal property of all probability distributions, whose normative desirability was emphasized by Bruno de Finetti [10], is called *quasi-additivity*,

$$p(A) \geqslant p(B) \Longleftrightarrow p(A\neg B) \geqslant p(B\neg A)$$

When strict atomic bound probabilities and linear possibilities agree for the mutually exclusive $A\neg B$ and $B\neg A$,

$$p(A) \geqslant p(B) \Longleftrightarrow \Pi(A\neg B) \geqslant \Pi(B\neg A) \quad (1)$$

Conversely, using a standard possibilistic relationship,

$$\Pi(A) \geqslant \Pi(B) \Longleftrightarrow \Pi(A) \geqslant \Pi(B\neg A)$$

we find, based upon agreement for exclusive sentences A and $B\neg A$,

$$\Pi(A) \geqslant \Pi(B) \Longleftrightarrow p(A) \geqslant p(B\neg A) \quad (2)$$

Results (1) and (2) say that orderings of strict atomic bound probabilities and those of linear possibility are syntactic restatements of one another. Any comparative thought expressed in one calculus has its corresponding expression in the other calculus. The translation in either direction can be accomplished mechanically. If one calculus is "reasonable" by some ordinal standard, then so is the other.

Since many normative arguments advanced by probabilists are ordinal in character, ordinal reasonableness should moot portrayals of possibility as normatively inferior to probability, or vice versa. That linear possibilities also solve the equations advanced in Cox's theorem [11], a pillar of probabilist normative argumentation, should secure the point.

Cox [12] argued further that sets of probability distributions, and not just individual measures, are reasonable representations of belief. If Cox's viewpoint is adopted, then all possibilities would be normatively acceptable in a probabilistic sense, not just the linear ones.

This expanded view of possibilistic reasonableness was discussed in [11], based on the observation that general possibilities syntactically describe sets of linear possibilities [9]. The orderings of general possibility distributions also have a direct syntactical relationship to the partial orderings expressed by general atomic bound solution sets.

A set of probabilities is said to express some ordinal relationship, e.g. A is ranked ahead of B, just when that order holds true in every distribution in the set, $p(A) > p(B)$. Suppose S is the solution set of a general atomic bound system. It can be shown that the only ordering assertions between distinct sentences which are displayed by every probability in S are strict orderings [8].

Suppose further that $\Pi(\ )$ is any possibility where, for all atoms a and b in the domain of the distributions belonging to S,

$$\pi(a) > \pi(b) \iff a >^* b$$
$$\pi(a) = \pi(b) \text{ otherwise}$$

In this section and the next, the relational operator "$>^*$" means a strict ordering assertion which is true in every distribution in S.

Since all probabilities are quasi-additive,

$$A >^* B \iff A\neg B >^* B\neg A$$

So, by similar arguments as were used to derive (1),

$$A >^* B \iff \Pi(A\neg B) > \Pi(B\neg A) \tag{3a}$$

Since possibility is completely ordered, and the alternative to $A >^* B$ or $B >^* A$ for distinct sentences is that A and B are unordered with respect to one another,

$$A \text{ unordered w.r.t. distinct } B \iff \Pi(A\neg B) = \Pi(B\neg A) \tag{3b}$$

For the expression of possibilistic orderings by the set S, we use the easily verified possibilistic relationship

$$\Pi(A) > \Pi(B) \iff \Pi(A\neg B) > \Pi(B)$$

to derive

$$\Pi(A) > \Pi(B) \iff A\neg B >^* B \tag{4a}$$

and since possibility is completely ordered,

$$\Pi(A) = \Pi(B) \iff \text{neither } A\neg B >^* B \text{ nor } B\neg A >^* A \tag{4b}$$

3. Necessity's static reasonableness alone and when breaking ties

Necessity's orderings also express comparative "thoughts" in possibilistic terms:

$$N(A) \geqslant N(B) \iff \Pi(\neg B) \geqslant \Pi(\neg A) \tag{5}$$

No special analysis is required to confirm the Cox-reasonableness of necessity, since in general any function of a Cox-reasonable belief representation is itself Cox-reasonable.

Necessity inherits linear possibility's ability to restate the strict atomic bound orderings and general possibility's restatement of the strict orderings in an atomic bound solution set. For necessity based on linear possibility, expressions (1), (2), (5), and DeMorgan's law imply

$$p(A) \geqslant p(B) \iff N(A \vee \neg B) \geqslant N(B \vee \neg A) \tag{6}$$

$$N(A) \geqslant N(B) \Longleftrightarrow p(A \vee \neg B) \geqslant p(B) \quad (7)$$

For necessity based upon general possibility, expressions (3a,b) and (4a,b) give

$$A >^* B \Longleftrightarrow \Pi(A\neg B) > \Pi(B\neg A) \Longleftrightarrow N(A \vee \neg B) > N(B \vee \neg A) \quad (8a)$$

$$A \text{ unordered w.r.t. distinct } B \Longleftrightarrow N(A \vee \neg B) = N(B \vee \neg A) \quad (8b)$$

$$N(A) > N(B) \Longleftrightarrow \Pi(\neg B) > \Pi(\neg A) \Longleftrightarrow A\neg B >^* \neg A$$
$$\Longleftrightarrow A >^* B \vee \neg A \quad (9a)$$

$$N(A) = N(B) \Longleftrightarrow \text{neither } A >^* B \vee \neg A \text{ nor } B >^* A \vee \neg B \quad (9b)$$

So, necessity orderings are fully expressive restatements of particular probabilistic orderings.

To discuss the reasonableness of necessity in possibilistic tie breaking, it is convenient to combine the two calculi into a single ordering,

$$A \text{ weakly precedes } B \Rightarrow \Pi(A) \geqslant \Pi(B)$$

$$A \text{ strictly precedes } B \Longleftrightarrow \Pi(A) > \Pi(B) \text{ or } \Pi(\neg B) > \Pi(\neg A)$$

A simple way to realize this ordering is to create a composite function, such as has been suggested in other contexts (e.g., in [2]),

$$f(A) = [\Pi(A) + N(A)]/2$$

This composition is possible since the necessity of all sentences whose possibility is less than unity is zero. No information about the possibilities and necessities is lost in computing $f(\ )$, since

$$f(A) < 1/2 \Rightarrow \Pi(A) = 2f(A); \quad N(A) = 0$$
$$\text{otherwise } \Pi(A) = 1; \quad N(A) = 2[f(A) - 1/2]$$

The ordering based on $f(\ )$ is transitive, and so linear or 'one dimensional.' It is sometimes said (e.g., [13]) that combining necessity and possibility offers two dimensions of information about uncertainties, i.e. credibility is valued by two numbers rather than one. Or, the two numbers might be used to define intervals which run from the necessity value to the possibility value for each sentence. Thus each interval includes zero, one, or both. Although such intervals can be arrayed linearly, they conspicuously differ from point-valued probability. Yet $f(\ )$ is point-valued and contains all the information, and just that information, which the two numbers do.

One dimension or two, which is it? Distinct information is offered by the two contributors to $f(\ )$. The two strands are linearly consonant, however. The $f(\ )$ ordering is not 'two dimensional' in any sense which prevents it from ordinally restating one-dimensional probabilistic orderings. As with possibility and necessity when used by itself, any comparative thought expressed by $f(\ )$

finds a syntactically corresponding expression in some probability orderings, and vice versa.

It is easily shown that no possibility ordering between mutually exclusive sentences can be changed by introducing necessity. It follows that

$$f(A\neg B) \geqslant f(B\neg A) \iff \Pi(A\neg B) \geqslant \Pi(B\neg A) \tag{10}$$

and from (1), in the special case of linear possibility and strict atomic bound probability,

$$p(A) \geqslant p(B) \iff f(A\neg B) \geqslant f(B\neg A)$$

In the other direction, we have

$$f(A) \geqslant f(B) \iff \Pi(A) \geqslant \Pi(B) \text{ and } N(A) \geqslant N(B)$$

and so, directly from (2) and (7):

$$f(A) \geqslant f(B) \iff p(A) \geqslant p(B\neg A) \text{ and } p(A \vee \neg B) \geqslant p(B)$$

The $f(\ )$ composition based on general possibility and necessity also restates the orderings which are unanimous in the solution set of an atomic bound system. Using (10) along with (3a),

$$A >^* B \iff f(A\neg B) > f(B\neg A)$$

and since $f(\ )$ is a complete ordering,

$$A \text{ unordered wrt distinct } B \iff f(A\neg B) = f(B\neg A)$$

For the expression of $f(\ )$ orderings by an atomic bound solution set's unanimous orderings, we have from the definition of $f(\ )$ along with (4a) and (9a),

$$f(A) > f(B) \iff A\neg B >^* B \text{ or } A\neg B >^* \neg A$$

with equality holding just when the conditions for both strict orderings of $f(\ )$ values are not met.

4. Necessity's enhancement of possibility's compliance with quasi-additivity

Cox appears not to have considered the normative status of single functions which combine two Cox-reasonable belief measures as $f(\ )$ does. He will not figure in our discussions of $f(\ )$. Considerations based upon de Finetti's quasi-additivity corroborate the impression of consensual reasonableness which the results of the previous section convey, and set the stage for discussing where the limits of consensus lie.

The tie-breaking $f(\ )$ ordering is a kind of quasi-additive extension of the strict orderings in the underlying possibility distribution. That is, for general (not just linear) possibilities,

$$f(A) > f(B) \Rightarrow \Pi(A\neg B) > \Pi(B\neg A) \tag{11}$$

To see this, we begin with a weakening of the definition of $f(\ )$, that

$$f(A) > f(B) \Rightarrow \Pi(A) > \Pi(B) \text{ or } N(A) > N(B)$$

and consider in turn each case of the disjunction.

For the possibilistic inequality, we have already encountered the possibilistic relationship

$$\Pi(A) > \Pi(B) \Rightarrow \Pi(A\neg B) > \Pi(B)$$

Since $B\neg A$ implies B, then by Łukasiewicz–Sugeno and transitivity,

$$\Pi(A) > \Pi(B) \Rightarrow \Pi(A\neg B) > \Pi(B\neg A) \tag{12}$$

For the ties which are broken by necessity, by substitution into (12),

$$\Pi(\neg B) > \Pi(\neg A) \Rightarrow \Pi(A\neg B) > \Pi(B\neg A)$$

Thus, by combining the two cases, we arrive at (11).

Necessity brings possibility closer to exhibiting a pattern of strict orderings which comports with the teachings of de Finetti. Moreover, a typical probabilist would agree with how every tie broken by necessity is resolved.

If $p(\ )$ is any single solution of a general atomic bound probability whose constraints agree with $\Pi(\ )$ in the strict orderings of the atoms, then from (11) and (3a),

$$f(A) > f(B) \Rightarrow p(A) > p(B)$$

In words, the strict orderings in the tie-breaking $f(\ )$ composition are a subset of the strict orderings in the related probabilistic orderings.

There is a limit to how fully necessity or anything else can achieve compliance with quasi-additivity without introducing other changes in the underlying possibilistic ordering, restricting atomic ties, or limiting the number of sentences. A complete transitive ordering which is based on *max* for mutually exclusive sentences cannot generally be quasi-additive for all sentence pairs.

For example, consider the ordering among four non-intersecting sentences

$$A =^* B >^* C >^* D$$

From here on in the paper, the symbols $\{=^*, \geqslant^*, >^*\}$ will denote relative position in any ordering of sentences, not just orderings based on sets of probabilities.

If quasi-additivity obtained in a *max* calculus, then we would have the cycle

$$\begin{aligned} A \vee C &=^* B \vee C \text{ (by quasi-additivity)} >^* B \vee D \text{ (by quasi-additivity)} \\ &=^* A \vee C \text{ (by } max\text{)} \end{aligned}$$

So, it is inevitable that necessity, like any general-purpose tie-breaking rule which retains the possibilistic order of exclusive sentences, would leave some ties unbroken.

5. A controversy about necessity's performance during belief revision

The specific possibilistic ties which necessity does and does not resolve lead to a breakdown in consensus when beliefs change. Even so, necessity's dynamic behavior may not be outside the range of views entertained within the probabilist community.

Consider four exclusive and exhaustive hypotheses whose possibilities are ranked

$$A >^* B >^* C >^* D$$

and A's possibility is one. Obviously, we have in a possibilistic ordering

$$B \vee C =^* B \vee D$$

which tie necessity cannot break because the liveliness of A forces a tie in necessity.

If we were later to learn that A is untrue, but the possibility ordering of the remaining hypotheses is unchanged, then we arrive at

$$B \vee C >^* B \vee D$$

in the new $f(\ )$ ordering and in necessity itself, since A no longer defeats the quasi-additive conclusion based on C's advantage over D.

There we encounter an impasse in intuitions. That the elimination of an alternative can change the credal order among the surviving hypotheses is acceptable to some. It is, for example, the nub of the famous "Peter, Paul, and Mary" case [2].

Many probabilists' intuition differs. Barring something special about how A was eliminated, its demise would leave the survivors in the same order as before in typical probabilist accounts of belief change. Since B, C, and D each imply $\neg A$, then in all probability distributions $p(\ )$, it is a standard result that

$$p(B \vee C \mid \neg A) > p(B \vee D \mid \neg A) \iff p(B \vee C) > p(B \vee D)$$

Only additional information beyond the mere elimination of A could alter the ordering among the surviving hypotheses for a typical probabilist.

It is somewhat ironic that if one simply leaves possibilistic ties intact, rather than breaking *some* of them in a consensually normative manner, then no dynamic controversy would arise. Conditioning can be defined within a possibilistic framework, as Dubois and Prade [14] have done, so that order among survivors is preserved when an alternative is eliminated.

Dynamic controversy does not contradict the findings of probabilistic reasonableness for necessity and for $f(\ )$ reported earlier. Commitment to a probability-resembling representation of static beliefs does not imply adoption of Bayesian conditioning as the exclusive mechanism of belief change [15]. Dempster's rule is one example [3], and Kyburg has favorably considered a liberal approach to belief change [16]. Both authors are easily identified as probabilists nevertheless.

The divergence of opinion between, and to some extent within, communities will not be settled here. When discussing consensus, one must acknowledge the boundary beyond which agreement is impossible. For necessity, that boundary abuts territory where probabilists are in some disagreement among themselves.

6. The Polya domains and "heuristic"

There are domains of uncertain reasoning in which necessity may experience another difficulty which can render it unable to serve as a possibilistic tie breaker. Interestingly, unaided possibility, ties and all, may be an especially attractive reasoning tool in these domains.

A pioneer explorer of the domains is the mathematician George Polya [17]. His principal concern was reasoning about the possible truth of mathematical conjectures, particularly as opinions in the matter change with the discovery of implied, analogous, or otherwise intuitively related facts. Polya noted that this domain was similar to the development of scientific theories and the determination of guilt in criminal investigations. Polya chose the name *heuristic* for the field of study which concerns the principles of reasoning in these domains.

Polya pursued his work from a consciously probabilistic perspective, freely drawing on both Bayesian and Keynesian predecessors, supplemented with some notions of his own. His resulting viewpoint was criticized by Bayesians, e.g. de Finetti [18].

As to necessity, belief change by hypothesis elimination is commonplace in these domains. Promising conjectures turn out to imply falsehoods, attractive theories are experimentally falsified, and chief suspects are exonerated by the discovery of unforeseen evidence. Opportunities for non-consensual dynamic behavior abound.

Furthermore, the negation of a hypothesis is often unavailable for credal evaluation with any useful specificity. For example, it is easy to think about the credal significance of finding a defendant's DNA at the scene of a crime in relation to the hypothesis that she is guilty. But what is "the" contrary hypothesis? Would that be "not guilty, and the defendant has innocent access to the scene," or "not guilty, and contamination occurred," or something else, perhaps something not yet even considered?

Both probability and possibility permit comparisons between sentences X and Y that do not depend upon specification of $\neg X$, $\neg Y$, nor $\neg X \neg Y$. This property, sometimes called "independence of irrelevant alternatives," is considered by some to be normatively desirable in its own right. In contrast, a usefully specific "not X" is required for the implementation of necessity.

The impracticality of specific negation would justify a divorce of possibility from necessity, at least in these domains. In itself, the negation problem would not forestall what might be called "contingent necessity," the complement of the possibility of all *known* alternatives. The lack of consensus surrounding hypothesis elimination and with those who prefer independence of irrelevant alternatives would persist, however.

There is also an affirmative argument for leaving all ties unbroken if possibility were applied within the Polya domains. A distinctive feature of the domains, not fully articulated by Polya himself, might be described as an ambivalence about the goals of inference. The mathematician, scientist, and jurist are all concerned with the discovery of truth. But that is not the whole of their jobs.

Polya sought to counsel mathematicians about what problems are worthwhile to work on. Discovering the truth of some implication of a conjecture not only encourages belief in the truth of the conjecture, but also establishes that the conjecture has consequences, that it might explain those consequences, that it is *interesting*.

Jurists self-consciously adopt rules of evidence which incorporate notions of fairness as well as probative value. A revealing hearsay may be ruled out of court not because it is uninformative, but because it would compromise a defendant's right to cross-examine adverse witnesses. The value to a prosecutor of the hearsay might combine its importance as an indicator of the truth of the charges along with its *usefulness*: whether or not the hearsay can fairly be introduced as part of a court case.

Scientists sometimes engage in especially subtle inferential episodes. Theories may be judged on their tractability and elegance along with their fidelity to the experimental record. Beauty may also be a factor in scientific thought [19], both as a value in its own right and as a heuristic guide to truth. As Roger Penrose described his own pattern of thinking (quoted in [20]):

> I have noticed on many occasions in my own work where there might, for example, be two guesses that could be made as to the solution of a problem and in the first case I would think how nice it would be if it were true; whereas in the second case I would not care very much about the result even if it were true.

Scientists' conclusions may also be more complicated than a simple statement of relative credibility. Newtonian mechanics survives in practice, despite its experimental falsification. Physicists simultaneously 'accept' incompatible theories, such as wave and particle models of light as the occasion demands, or tolerate the unresolved discrepancies between relativity and quantum mechanics.

Throughout the Polya domains, then, a "good" conclusion is not necessarily determined by beliefs about the simple truth of the matter. The merit ascribed to a hypothesis may reflect preference (interestingness, usefulness, fairness, ...) unrebutted by the evidence, rather than simply an assessment of likely truth impelled by the evidence and prior knowledge.

If this characterization of expert goal-setting practice is accurate, then there may be a role for an inferential calculus which serves both for reasoning about preferences as well as for ordinary reasoning about credibility in the style of Łukasiewicz–Sugeno.

7. A distinctive feature of the possibility calculus

The value of the Łukasiewicz–Sugeno insight is that it captures much of the intuitive force of what people mean when they speak of credibility, while preserving a high degree of generality. Suppose one set out to characterize reasoning about preferences with a similar goal.

In a domain of mutually exclusive rewards, we see immediately that reasoning about preferences is conspicuously unlike Łukasiewicz–Sugeno. Offered the choice between

a commitment to be paid \$5

a commitment to be paid \$5 or else \$1

the stronger, rather than the weaker, commitment may well be preferred. If the issue were credibility, perhaps as judged by an onlooker wondering how much money will change hands, then Łukasiewicz–Sugeno never disfavors the weaker sentence.

But preference does not always follow logical strength. Between

a commitment to be paid \$1

a commitment to be paid \$5 or else \$1

considerations of dominance at least weakly favor the weaker commitment in this case.

This easy, homely example suggests one candidate for a general abstract description of a preference ordering among sentences describing outcomes:

$$\text{if } A \Rightarrow B, \quad \text{then } B >^* A \text{ implies } B\neg A >^* A \tag{13a}$$

or equivalently in a complete ordering,

$$\text{if } A \Rightarrow B, \quad \text{then } A \geqslant^* B\neg A \ \text{ implies } \ A \geqslant^* B \tag{13b}$$

As with Łukasiewicz–Sugeno, we would expect more from a practical calculus, but it is plausible that we would not be content with less.

Although derived from an elementary observation about preference, relationship (13a,b) states one ordinal property of the ordinary Boolean logic, as Łukasiewicz–Sugeno states another. The relationship also expresses a property displayed by orderings of evidentiary support using ordinary conditional probabilities. It is easily verified that for all probabilities $p(\ )$,

$$\text{if } A \Rightarrow B, \quad \text{then } p(e \mid B) > p(e \mid A) \ \text{ implies } \ p(e \mid B\neg A) > p(e \mid A)$$

That preference and conditional probabilities are so closely related is unsurprising. Borch [21] showed that some expected utility models of preference compute a probability of economic ruin conditioned upon a chosen act. All conventional expected utility models can be viewed as computing a tight upper bound on a similar conditional probability [22].

In comparing the two kinds of ordering criteria

$$\text{Łukasiewicz–Sugeno}: \quad \text{if } A \Rightarrow B, \quad \text{then } B \geqslant^* A$$

$$\text{preference-support}: \quad \text{if } A \Rightarrow B, \quad \text{then } A \geqslant^* B\neg A \ \text{ implies } \ A \geqslant^* B$$

it is straightforward that a *max* rule for disjunctions satisfies *both* criteria. *Max* is the only rule which relates sentences according to both criteria in a transitive ordering.

Proposition. *If* $\{>^*, =^*\}$ *is a transitive, Łukasiewicz–Sugeno, and preference-support ordering in which all equivalent sentences are ranked equally, then if* a *is a top atom in sentence* A*, then* $a =^* A$.

Proof. If a is the only atom in A, then a is equivalent to A, and $a =^* A$. Otherwise, let b be any atom in A other than a. By Łukasiewicz–Sugeno, $a \vee b \geqslant^* a$. Since a is a top atom, $a \geqslant^* b$, thus $a \geqslant^* a \vee b$ by preference-support, so $a =^* a \vee b$. Similarly, if $a =^* a \vee C$, and $a \geqslant^* d$, then $a =^* a \vee C \vee d$, and so we can construct the obvious induction over all the atoms in A, leading to $a =^* A$. $\square$

Possibility, the *max* calculus, stands alone therefore as the only completely ordered calculus that is both Łukasiewicz–Sugeno and preference-support. Any mechanism for tie breaking which acted on implications would force the new combined calculus to be either Łukasiewicz–Sugeno alone, or preference-support alone, or perhaps neither. The ties are how the possibility calculus manages to walk the tightrope between the all-but-conflicting criteria.

Possibility's twin aspect combining reasoning about credibility and about preference or support may offer a promising vehicle for investigations in

Polya's realm of incompletely formulated alternatives and ambiguous inferential goals. It may even provide a useful alternate formulation of Polya's account of the territory. That possibility and probability are so closely related suggests that such an alternative formulation could retain much of the intuitive and normative force of Polya's original exposition.

During the course of a 1954 rebuttal to de Finetti [16, Chapter XV, Section 7], Polya speculated about the prospects for exploiting infinitesimal expressions within his heuristic. He did not develop this approach in much detail. However, the by-now familiar relationships among default reasoning, infinitesimal probabilities, standard probabilities, and possibility strongly suggest that Polya might have been open to a possibilistic exploration, had that calculus been available to him.

8. Conclusions

The normative case in favor of necessity, both in its own right and as a tie-breaking mechanism for possibility, is considerable. There is, however, a principled disagreement among scholars when belief changes by hypothesis elimination. Both necessity and the combination of necessity with possibility falls on one side of this divide. Possibility itself, in some interpretations of its conditioning behavior, can be placed on the other side along with typical interpretations of probabilistic conditioning.

Unaided possibility seems well-suited for exploring interesting and important domains which engaged George Polya. Those domains' reliance on hypothesis elimination and their hostility to evaluation based on complementation diminish the appeal of necessity there.

Another feature of Polya's domains suggests that possibility's ties might best be left unbroken. Merit within the domains is often a combination of both credibility and preference. Credibility orderings are frequently defined by reference to an intuitive criterion suggested by Łukasiewicz and revived by Sugeno. A comparably general criterion is proposed here to characterize preference orderings and those based upon evidentiary support.

Possibility and only possibility, unassisted by any mechanism for tie breaking, gives rise to a complete ordering which represents both credibility and preference according to these criteria. Possibility's ties support its ability to mirror the ambiguity of merit within Polya's domains. On some occasions, then, the use of possibility without necessity may be attractive.

References

[1] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3–28.

[2] D. Dubois, H. Prade, P. Smets, Representing partial ignorance, IEEE Transactions on System, Man, and Cybernetics 26 (1996) 361–377.

[3] A.P. Dempster, A generalization of Bayesian inference, Journal of the Royal Statistical Society B 30 (1968) 205–247 (with discussion).

[4] R.T. Cox, Probability, frequency, and reasonable expectation, American Journal of Physics 14 (1946) 1–13.

[5] J. Łukasiewicz, Logical foundations of probability theory, in: J. Borkowski (Ed.), Jan Łukasiewicz, Selected Works, North-Holland, Amsterdam, 1970.

[6] M. Sugeno, Theory of Fuzzy Integral and its Applications, Ph.D. Dissertation, Tokyo Institute of Technology, 1974, unpublished, cited in H. Prade, A computational approach to approximate and plausible reasoning with applications to expert systems, IEEE Transactions on Pattern Analysis and Machine Intelligence 7 (1985) 260–283.

[7] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic logic, preferential models and cumulative logics, Artificial Intelligence 44 (1990) 167–207.

[8] P. Snow, Diverse confidence levels in a probabilistic semantics for conditional logics, Artificial Intelligence 113 (1999) 269–279.

[9] S. Benferhat, D. Dubois, H. Prade, Possibilistic and standard probabilistic semantics of conditional knowledge, in: Proceedings of the AAAI, AAAI Press, Menlo Park, CA, 1997, pp. 70–75.

[10] B. de Finetti, La prévision, ses lois logiques, ses sources subjectives, Annales de l'Institut Henri Poincaré, 1937; English translation by H.E. Kyburg Jr., in: H.E. Kyburg, H. Smokler (Eds.), Studies in Subjective Probability, John Wiley, New York, 1964.

[11] P. Snow, The reasonableness of possibility from the perspective of Cox, Computational Intelligence 17 (2001) 178–192.

[12] R.T. Cox, The Algebra of Probable Inference, Johns Hopkins Press, Baltimore, 1961 (Chapter 6).

[13] D. Dubois, H. Prade, Fuzzy sets—a convenient fiction for modeling vagueness and possibility, IEEE Transactions on Fuzzy Systems 2 (1994) 16–21.

[14] D. Dubois, H. Prade, Possibilistic inference under matrix form, in: H. Prade, C.V. Negoita (Eds.), Fuzzy Logic in Knowledge Engineering, Springer TUV, Rhineland, 1986, pp. 112–126.

[15] P. Snow, Associativity and normative credal probability, IEEE Transactions on Systems, Man, and Cybernetics 32B (2002) 4–12.

[16] H. Kyburg, Subjective probability: criticism, reflection, and problems, Journal of Philosophical Logic 7 (1978) 157–180.

[17] G. Polya, Mathematics and Plausible Reasoning, Princeton University Press, Princeton, New Jersey, 1954.

[18] B. de Finetti, La 'logica del plausibile' secondo la concezione di Polya, in: Atti della XLII Riunione della Societa Italiana per il Progresso delle Scienze, 1951, pp. 227–236.

[19] J.W. McAllister, Is beauty a sign of truth in scientific theories?, American Scientist 86 (1998) 174–183.

[20] E. Klarreich, Foams and honeycombs, American Scientist 88 (2000) 152–161.

[21] K. Borch, The theory of risk, Journal of the Royal Statistical Society 29 (1967) 432–467.

[22] P. Snow, Maximizing expected utility is a survival criterion, Theory and Decision 22 (1987) 143–154.